

On the Series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n f^{(n)}(t)$$

S.E.Akrami*

Departments of Mathematics and Physics,
Institute for Research in Fundamental Sciences,
P.O.Box:19395-5746, Tehran, Iran

January 20, 2013

Abstract

We study the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n f^{(n)}(t)$. We show that for analytic functions this series is uniformly and absolutely convergent to the constant $f(0)$. We show that there are nowhere analytic functions for them the series is divergent for all t and also there are nowhere analytic functions for them the series is convergent to $f(0)$ at least for t in a dense subset of \mathbb{R} .

Acknowledgement I thank Allah Rabbel-Alamin and Imam Zaman.

1 Introduction

This paper is about the convergence of the series

$$\hat{f}(t) := \sum_{n=0}^{\infty} (-1)^n \frac{f^{(n)}(t)}{n!} t^n. \quad (1.1)$$

*Supported financially by the grant 83810319 from Institute for Research in Fundamental Sciences, P.O.Box:19395-5746, Tehran, Iran. E-mail:akramisa@ipm.ir

First notice that if we differentiate the series term by term we get

$$\begin{aligned}
\frac{d}{dt}\hat{f}(t) &= \sum_{n=0}^{\infty}(-1)^n\frac{f^{(n+1)}(t)}{n!}t^n + \sum_{n=1}^{\infty}(-1)^n\frac{f^{(n)}(t)}{(n-1)!}t^{n-1} \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{f^{(n+1)}(t)}{n!}t^n - \sum_{n=0}^{\infty}(-1)^n\frac{f^{(n+1)}(t)}{n!}t^n \\
&= 0
\end{aligned}$$

Thus $\hat{f}(t)$ should be constant. But the point is that we are not allowed to differentiate term by term from a series even the series is uniformly convergent. We shall show that for analytic functions around origin this series is convergent to the constant $f(0)$. The surprising point here is not the proof which is very easy, but it is very strange that why this fact has been forgotten in textbooks of mathematical analysis! At least it might be mentioned in them as an exercise. However I could not find any trace of this strange series in the mathematics literature, I could see it in quantum mechanics textbooks, see for example [1]. There, it is named as the translation operator. Physicists define an operator as

$$(T_a f)(t) := f(t + a). \quad (1.2)$$

Then they claim that this operator is equal to the operator

$$e^{a\frac{d}{dt}} = \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{d^n}{dt^n}. \quad (1.3)$$

But in fact to prove this they implicitly or sometimes explicitly assume that f is analytic. In fact one can easily prove, as we shall prove in next section, that for analytic functions these two operators coincide. Notice that if we set $a = -t$ we get our strange series.

Some textbooks,[1], try to prove the equality of the operators (1.2) and (1.3) as follows. They first expand f as a Fourier integral

$$f(t) = \int_{-\infty}^{\infty} a(x)e^{ixt}dx, \quad (1.4)$$

and then apply the operator (1.3) to this integral. They exchange the order

of the derivative $\frac{d^n}{dt^n}$ with the integral.

$$\begin{aligned}
e^{-t\frac{d}{dt}}f &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n f^{(n)}(t) \\
&= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{(-1)^n}{n!} (ix)^n t^n a(x) e^{ixt} dx \\
&= \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(-ixt)^n}{n!} a(x) e^{ixt} dx \\
&= \int_{-\infty}^{\infty} e^{-ixt} a(x) e^{ixt} dx \\
&= \int_{-\infty}^{\infty} a(x) dx \\
&= f(0)
\end{aligned}$$

But in the third line of the above proof we have exchanged the order of integral and summation, but this is not allowed unless the series $\sum_{n=0}^{\infty} \frac{(-ixt)^n}{n!} a(x) e^{ixt} = a(x)$ is uniformly convergent. But this is not always true and depends to the coefficient $a(x)$. For example if $a(x)$ has compact support then the series is uniformly convergent.

2 Convergence Tests

Proposition 1 *A necessary condition for the point-wise (uniformly) convergence of the series \hat{f} is that*

$$\lim_{n \rightarrow \infty} \frac{1}{n!} t^n f^{(n)}(t) = 0, \quad (2.1)$$

point-wise (uniformly).

Proof A necessary condition for the convergence of a series $\sum_n a_n$ is $\lim_{n \rightarrow \infty} a_n = 0$. ■

Theorem 2 *Let for some $k > 0$ the limit*

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n!} t^n f^{(n+k)}(t) \quad (2.2)$$

exists uniformly around origin. If $k > 1$ then the series \hat{f} is uniformly convergent to the constant $f(0)$ and if $k = 1$ then the series \hat{f} is uniformly convergent and differentiable. If when $k = 1$ we denote the limit (2.2) by $\check{f}(t)$, then we have $\frac{d}{dt} \hat{f}(t) = \check{f}(t)$. In particular case when $\check{f}(t) = 0$ for all t , i.e.

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n!} t^n f^{(n+1)}(t) \rightarrow 0 \quad (2.3)$$

uniformly, we get $\hat{f}(t) = f(0)$.

Proof Let S_N be the partial sum of the series obtained by the term by term differentiation of the series \hat{f} . Thus

$$\begin{aligned} S_N &= \sum_{n=0}^N \frac{(-1)^n}{n!} t^n f^{(n+1)}(t) + \sum_{n=1}^N \frac{(-1)^n}{(n-1)!} t^{n-1} f^{(n)}(t) \\ &= \sum_{n=0}^N \frac{(-1)^n}{n!} t^n f^{(n+1)}(t) - \sum_{n=0}^{N-1} \frac{(-1)^n}{n!} t^n f^{(n+1)}(t) \\ &= \frac{(-1)^N}{N!} t^N f^{(N+1)}(t). \end{aligned}$$

Now let us define

$$f_{k,n}(t) := \frac{(-1)^n}{n!} t^n f^{(n+k)}(t). \quad (2.4)$$

We have $f'_{k-1,n}(t) = f_{k,n}(t) - f_{k,n-1}(t)$. Thus if for some $k > 1$, $\lim_{n \rightarrow \infty} f_{k,n}(t)$ exists uniformly then $\lim_{n \rightarrow \infty} f'_{k-1,n}(t) = 0$ uniformly. Thus $\lim_{n \rightarrow \infty} f_{k-1,n}(t)$ exists uniformly and is constant and since we have $f_{k,n}(0) = 0$ for all k we deduce that $\lim_{n \rightarrow \infty} f_{k-1,n}(t) = 0$ uniformly. By repeating this argument we deduce that $\lim_{n \rightarrow \infty} f_{1,n}(t) = 0$ uniformly. But we have $f_{1,N}(t) = S_N$. Thus the series obtained by the term by term differentiation of the series \hat{f} is uniformly convergent and since the series \hat{f} is convergent at $t = 0$, then by a well known theorem in mathematical analysis (see [2] for example) the series \hat{f} is also uniformly convergent and differentiable and $\frac{d}{dt} \hat{f}(t) = \lim_{N \rightarrow \infty} S_N = 0$. Thus $\hat{f}(t) = f(0)$.

Now if $k = 1$ then $\lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} f_{1,N}(t)$ exists uniformly but not necessarily vanishes. Thus the series obtained by the term by term differentiation of the series \hat{f} is uniformly convergent and therefore the series \hat{f} is also uniformly convergent and differentiable and $\frac{d}{dt} \hat{f}(t) = \check{f}(t)$. ■

Theorem 3 *If there exists constants C and M such that*

$$|f^{(n)}(t)| < CM^n \quad (2.5)$$

around origin. Then the series \hat{f} is uniformly convergent to the constant $f(0)$.

Proof Let $t \in (-a, a)$ for some a . Then we have $|\frac{(-1)^n}{n!} t^n f^{(n+1)}(t)| \leq CM \frac{(Ma)^n}{n!}$. But it is easy to show that the right hand side of this inequality goes to zero. Thus we can use the theorem 2. ■

Theorem 4 *If there exists a constant M such that $|\frac{f^{(n+1)}(t)}{f^{(n)}(t)}| < M$ around origin. Then the series \hat{f} is uniformly convergent to the constant $f(0)$.*

Proof Let $t \in [-a, a]$ for some a . We have $|f^{(n+1)}(t)| < M|f^{(n)}(t)|$. Thus $|f^{(n)}(t)| < M^n |f(t)|$. Now let C be the maximum of f at $[-a, a]$. Thus $|f^{(n)}(t)| < CM^n$. Now use the theorem 3. ■

Theorem 5 *If there exists a constant M such that $|\sum_{n=0}^N (-1)^n f^{(n)}(t)| < M$ for all N uniformly around origin. Then the series \hat{f} is uniformly convergent but not necessarily to a constant.*

Proof We use the Dirichlet' test (see [2] for example) which states that if there exists a constant M such that $|\sum_{n=0}^N f_n(t)| < M$ for all N uniformly and $g_n(t)$ is a decreasing sequence converging to zero uniformly, then the series $\sum f_n(t)g_n(t)$ is uniformly convergent. In our case we set $f_n(t) := (-1)^n f^{(n)}(t)$ and $g_n(t) := t^n/n!$. ■

Theorem 6 *Let*

$$\alpha := \sup_t \limsup_n \sqrt[n]{\frac{|f^{(n)}(t)|}{n!}}, \quad \beta := \inf_t \limsup_n \sqrt[n]{\frac{|f^{(n)}(t)|}{n!}}, \quad (2.6)$$

and $R := \frac{1}{\alpha}$, $S := \frac{1}{\beta}$. Then the series $\hat{f}(t)$ for $|t| < R$ is absolutely convergent but not necessarily to $f(0)$ and for $|t| > S$ is divergent.

Proof Let $a_n(t) := \frac{(-1)^n}{n!} f^{(n)}(t)$. Thus $\hat{f}(t) = \sum_{n=0}^{\infty} a_n(t)t^n$. Now for $|t| < R$ we have $\limsup_n \sqrt[n]{|a_n(t)t^n|} = |t| \limsup_n \sqrt[n]{\frac{|f^{(n)}(t)|}{n!}} \leq \alpha|t| < 1$. Thus the series $\sum_{n=0}^{\infty} a_n(t)t^n = \hat{f}(t)$ is absolutely convergent. The other part is similar. ■

Proposition 7 (i) *If the series \hat{f} and \hat{g} are either point-wise or uniformly and either absolutely or conditional convergent then the series $\widehat{af + bg}$ are so and $\widehat{af + bg} = a\hat{f} + b\hat{g}$*

(ii) *If at least one of series \hat{f} and \hat{g} are absolutely (either point-wise or uniformly) convergent then the series \widehat{fg} is also (either point-wise or uniformly) convergent and $\widehat{fg} = \hat{f}\hat{g}$.*

Proof (i) is obvious. (ii) Using the fact that $(fg)^{(n)} = \sum_{i=0}^n \frac{n!}{i!(n-i)!} f^{(i)} g^{(n-i)}$ we have

$$\begin{aligned} \hat{f}(t)\hat{g}(t) &= \sum_{n=0}^{\infty} (-1)^n \frac{f^{(n)}(t)}{n!} t^n \sum_{n=0}^{\infty} (-1)^n \frac{g^{(n)}(t)}{n!} t^n \\ &= \sum_{n=0}^{\infty} \sum_{i+j=n} (-1)^i \frac{f^{(i)}(t)}{i!} (-1)^j \frac{g^{(j)}(t)}{j!} t^i t^j \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^n \frac{n!}{i!(n-i)!} f^{(i)}(t) g^{(n-i)}(t) \frac{(-1)^n}{n!} t^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(fg)^{(n)}(t)}{n!} t^n \\ &= \widehat{fg}(t) \end{aligned}$$

Where we used the Mertens' theorem about the convergence of the Cauchy's product of two series [2]. ■

Proposition 8 *If $g(t) := f(at)$ for some constant a . Then $\hat{g}(t) = \hat{f}(at)$.*

Proof $\hat{g}(t) := \sum_{n=0}^{\infty} (-1)^n \frac{g^{(n)}(t)}{n!} t^n = \sum_{n=0}^{\infty} (-1)^n \frac{f^{(n)}(at)}{n!} a^n t^n = \hat{f}(at)$. ■

Theorem 9 *If the series \hat{f} is uniformly convergent (not necessarily to a constant) then for the anti-derivative of f , i.e. $F(x) = \int_0^x f(t)dt$ the series \hat{F} is uniformly convergent to the constant $F(0) = 0$.*

Proof First, notice that if we set $f_n(x) := \int_0^x t^n f^{(n)}(t)dt$ then by the integration by parts we get $f_n(x) := x^n f^{(n-1)}(x) - n f_{n-1}(x)$. Thus $x^n F^{(n)}(x) = x^n f^{(n-1)}(x) = f_n(x) + n f_{n-1}(x)$. Now since \hat{f} is a uniformly convergent series of integrable functions, we can integrate term by term, [2]. Thus

$$\begin{aligned} \int_0^x \hat{f}(t)dt &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^n f^{(n)}(t)dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} f_n(x) \end{aligned}$$

Next we get

$$\begin{aligned} \hat{F}(x) &= F(x) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^n F^{(n)}(x) \\ &= F(x) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} f_n(x) - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} f_{n-1}(x) \\ &= F(x) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} f_n(x) - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} f_n(x) \\ &= F(x) - \int_0^x f(t)dt \\ &= 0 \end{aligned}$$

■

Theorem 10 *If a function f can be expanded around origin via power series $f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n$, $|t| < R$, then \hat{f} is absolutely and uniformly convergent to constant $f(0)$ in the interval $|t| < R/2$. That is*

$$\sum_{n=0}^{\infty} (-1)^n \frac{f^{(n)}(t)}{n!} t^n = f(0), \quad |t| < R/2, \quad (2.7)$$

for analytic functions. If $R = \infty$ then $\hat{f}(t) = f(0)$ for all $t \in \mathbb{R}$.

Proof It is known that (see [2] for example) for any x satisfying $|x| < R$, we can expand f around x as $f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (t-x)^n$. This holds for all t satisfying $|t-x| < R-|x|$. Now if $|x| < R/2$ then $|0-x| < R-|x|$. Thus we can put $x = 0$ in the last series to get $f(0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (-x)^n = \hat{f}(x)$. ■

The following example shows that we can not make larger the domain of validity of $\hat{f}(x) = f(0)$ from the domain $|x| < R/2$.

Example 11 For the function $f(t) = \frac{1}{1+t}$ the series \hat{f} is absolutely and uniformly convergent to the constant $f(0)$ for $\frac{-1}{2} < t$ and otherwise is divergent.

Proof Using induction on n we have $f^{(n)}(t) = (-1)^n n! \frac{1}{(1+t)^{n+1}}$. Thus

$$\begin{aligned} \hat{f}(t) &= \sum_{n=0}^{\infty} \frac{t^n}{(1+t)^{n+1}} \\ &= \frac{1}{1+t} \sum_{n=0}^{\infty} \left(\frac{t}{1+t}\right)^n \\ &= \frac{1}{1+t} \frac{1}{1 - \frac{t}{1+t}} \\ &= 1 \\ &= f(0). \end{aligned}$$

The convergence holds when $|\frac{t}{1+t}| < 1$. i.e when $\frac{-1}{2} < t$. ■

Theorem 12 Let f be a smooth function. Let $t \neq 0$ belongs to its domain. Consider the Taylor series around t

$$(T_t f)(x) := \sum_{n=0}^{\infty} \frac{f^{(n)}(t)}{n!} (x-t)^n, \quad (2.8)$$

whose radius of convergence is denoted by R_t .

(i) If 0 does not belong to the convergence interval of (2.8), i.e. if $R_t < |t|$, in particular if $R_t = 0$, then $\hat{f}(t)$ diverges.

(ii) If 0 belongs to the convergence interval of (2.8), i.e. if $R_t > |t|$, then $\hat{f}(t)$ converges to $(T_t f)(0)$. Moreover if f is analytic at t , i.e. if $T_t f = f$ then $\hat{f}(t)$ converges to $f(0)$.

(iii) If 0 belongs to the boundary of the convergence interval of (2.8), i.e. if $R_t = |t|$, then in general one can not say anything about the convergence of $\hat{f}(t)$. But if we know priory that $(T_t f)(x)$ converges at $x = 0$ or equivalently $\hat{f}(t)$ is convergent then the limit $\lim_{x \rightarrow 0} (T_t f)(x)$ exists and is equal to $\hat{f}(t)$. In particular if f is analytic at t and moreover 0 belongs to the domain of f then $\hat{f}(t) = f(0)$.

Proof Since indeed $\hat{f}(t) = (T_t f)(0)$ the cases (i) and (ii) are clear. Case (iii) is the Able's theorem [2]. ■

Remark 1) The example $f(t) = \frac{1}{1+t}$ which was investigated above using theorem (10) can also be studied by the theorem (12) with the same results.

2) This theorem gives us another proof for the theorem (10).

Now we study the series $\hat{f}(t)$ for some examples of nonanalytic functions. The first example is about a function which is analytic everywhere except at $t = 0$

Example 13 *For the function*

$$f(t) := e^{\frac{-1}{t^2}}, \quad t \neq 0, \quad f(0) = 0. \quad (2.9)$$

If $\hat{f}(t)$ is convergent then we must have $\hat{f}(t) = f(0)$. But unfortunately until the time of writing of this article we do not know anything about the convergence of $\hat{f}(t)$ for $t \neq 0$.

Proof If f is analytic at some $t \neq 0$ in which $R_t < |t|$ then by a well known theorem of mathematical analysis [2], f must be analytic at 0. But it is a well known fact that f is not analytic at 0. Thus we must have $R_t \geq |t|$. In fact this function satisfies the case (iii). That is f is analytic at any $t \neq 0$ and we have $R_t = |t|$. Because this function is the composition of a everywhere analytic function $g(t) = e^t$ and the function $h(t) = \frac{-1}{t^2}$ which is analytic everywhere except at $t = 0$ and we know that for such a composite $f = g \circ h$ function in which g is analytic everywhere, the radius of convergence of f at t is equal to the radius of convergence of h at t . And clearly the radius of convergence of h at $t \neq 0$ is $R_t = |t|$. ■

Example 14 *Let u be a smooth periodic positive function with period ℓ , analytic at each point except at points $m\ell$ where $m \in \mathbb{Z}$, for each n there exists a constant M_n such that $|u^{(n)}(x)| < M_n$ for all x and $u^{(n)}(0) = 0$ for all n . Let a_n be a sequence of positive real numbers such that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges all over \mathbb{R} . Then the following function*

$$f(x) = \sum_{n=0}^{\infty} a_n u(2^n x), \quad (2.10)$$

is smooth but nowhere analytic. That is at each point a the Taylor series at a either diverges or does not converge to $f(x)$.

Moreover $\hat{f}(t)$ for all $t \in \{\frac{(2m+1)\ell}{2^n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$ is convergent to $f(0) = 0$. In other points we do not know anything on the convergence of $\hat{f}(t)$ and its values.

Proof Since $|a_n u(2^n x)| < a_n M_0$ and since the series $\sum_n a_n$ converges, the series (2.10) is uniformly convergent and thus f is well defined. Moreover

we have $|\frac{d^k}{dx^k}(a_n u(2^n x))| \leq |2^{kn} a_n u^{(k)}(2^n x)| < 2^{kn} a_n M_k$ and since the series $\sum_n a_n (2^k)^n$ is convergent by hypothesis, we conclude that the series $\sum_n \frac{d^k}{dx^k}(a_n u(2^n x))$ is uniformly convergent and thus f is smooth.

Next suppose that f is analytic at some point. Then since analyticity at a point implies analyticity at some neighborhood of that point and since the set of numbers of the form $\frac{m\ell}{2^n}$ where m is an odd integer and n is a natural number, is dense in \mathbb{R} , we may assume that f is analytic at $a = \frac{m_0\ell}{2^{n_0}}$ for some m_0 and n_0 . Now let $g(x) := \sum_{n=0}^{n_0} a_n u(2^n x)$, $h(x) := \sum_{n=n_0+1}^{\infty} a_n u(2^n x)$. Thus $h = f - g$ and since g is analytic at a then analyticity of f at a implies analyticity of h at a . But we have $h^{(k)}(a) = \sum_{n=n_0+1}^{\infty} 2^{kn} a_n u^{(k)}(2^n \frac{m_0\ell}{2^{n_0}}) = \sum_{n=n_0+1}^{\infty} 2^{kn} a_n u^{(k)}(2^{n-n_0} m_0\ell) = 0$, since periodicity of u implies that $u^{(k)}(2^{n-n_0} m_0\ell) = 0$. Thus h should vanish around a but this is a contradiction by hypothesis of positiveness of the sequence a_n and the function u .

We showed that at each point of the form $a = \frac{m_0\ell}{2^{n_0}}$ for some odd integer m_0 and natural number n_0 , the Taylor series at a converges to $g(x)$ for all $x \in \mathbb{R}$. Thus by the theorem (12) part (ii), $\hat{f}(\frac{m_0\ell}{2^{n_0}})$ is well defined and is equal to $g(0)$. But we have $g(0) = f(0) = 0$. ■

The following example is from [3].

Example 15 According to the example 14, let $a_n := \frac{1}{n!}$ and $u(x) := \beta(x - [x])$. Where $\beta(x)$ is any smooth function on $[0, 1]$ which is positive and analytic on $(0, 1)$ and $\beta^{(n)}(0) = \beta^{(n)}(1) = 0$ for all n , and for each n there exists a constant M_n such that $|\beta^{(n)}(x)| < M_n$ for all x . For instance let $\beta(x) := \alpha(x)\alpha(1-x)$ where $\alpha(x)$ is any smooth function which is positive and analytic on $(0, 1)$ and $\alpha^{(n)}(0) = 0$ for all n and for each n there exists a constant M_n such that $|\alpha^{(n)}(x)| < M_n$ for all x . For instance let $\alpha(x) := e^{\frac{-1}{x^s}}$, $\alpha(0) = 0$ where $s = 1$ or $s = 2$. Thus the functions

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \beta(2^n x - [2^n x]), \quad (2.11)$$

are smooth but nowhere analytic. Thus $\hat{f}(t)$ for all $t \in \{\frac{(2m+1)}{2^n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$ is convergent to $f(0) = 0$. In other points we do not know anything on the convergence of $\hat{f}(t)$ and its values.

Proof We show u is smooth at every point a . If a is not an integer then $x - [x]$ being equal to the smooth function $x - [a]$, around a , is smooth and therefore since β is smooth, the composition $u(x) = \beta(x - [a])$ is smooth at a and $u^{(k)}(a) = \beta^{(k)}(a - [a])$. Next let $a = n$ be an integer. We have $u'(n+) = \lim_{x \rightarrow n+} \frac{u(x) - u(n)}{x - n} = \lim_{x \rightarrow n+} \frac{\beta(x - n) - \beta(0)}{x - n} = \beta'(0) = 0$, and $u'(n-) = \lim_{x \rightarrow n-} \frac{u(x) - u(n)}{x - n} = \lim_{x \rightarrow n-} \frac{\beta(x - n + 1) - \beta(1)}{x - n} = \beta'(1) = 0$. Thus $u'(a) = 0$. Now let $u^{(k)}(n)$ exists and is equal to zero. We have $u^{(k+1)}(n+) = \lim_{x \rightarrow n+} \frac{u^{(k)}(x) - u^{(k)}(n)}{x - n} = \lim_{x \rightarrow n+} \frac{\beta^{(k)}(x - n) - \beta^{(k)}(0)}{x - n} = \beta^{(k+1)}(0) =$

0 and similarly $u^{(k+1)}(n-) = \lim_{x \rightarrow n-} \frac{u^{(k)}(x) - u^{(k)}(n)}{x-n} = \lim_{x \rightarrow n-} \frac{\beta^{(k)}(x-n+1)}{x-n} = \lim_{x \rightarrow n-} \frac{\beta^{(k)}(x-n+1) - \beta^{(k)}(1)}{x-n} = \beta^{(k+1)}(1) = 0$. Thus $u^{(k+1)}(n) = 0$. Therefore u is smooth.

Clearly u is periodic with period 1 and analytic at everywhere except at integers. Also we have $|u^{(k)}(x)| = |\beta^{(k)}(x - [x])| < M_n$.

When $\beta(x) := \alpha(x)\alpha(1-x)$, by the Leibnitz rule

$$\beta^{(k)}(x) = \sum_{k=0}^n (-1)^{n-k} \frac{n!}{k!(n-k)!} \alpha^{(k)}(x) \alpha^{(n-k)}(1-x)$$

and the fact that $\alpha^{(k)}(0) = 0$ we conclude that $\beta^{(k)}(0) = \beta^{(k)}(1) = 0$. Also $|\beta^{(k)}(x)| \leq \sum_{k=0}^n \frac{n!}{k!(n-k)!} M_k M_{n-k}$. Thus derivatives of β are bounded.

One can easily check that the functions $\alpha(x) := e^{\frac{-1}{x^s}}$, $\alpha(0) = 0$ where $s = 1$ or $s = 2$ satisfy all the required conditions. ■

The following example is from [4].

Example 16 According to the example 14, let $u(x) := \alpha(\sin x)$ where $\alpha(x) := e^{\frac{-1}{x^2}}$, $x \neq 0$, $\alpha(0) = 0$ and let $a_n := 2^{-2^n}$. Thus the functions

$$f(x) = \sum_{n=0}^{\infty} 2^{-2^n} e^{-\csc^2(2^n x)}, \quad (2.12)$$

is smooth but nowhere analytic. Thus $\hat{f}(t)$ for all $t \in \{\frac{2\pi(2m+1)}{2^n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$ is convergent to $f(0) = 0$. In other points we do not know anything on the convergence of $\hat{f}(t)$ and its values.

Proof Since $u^{(n)}(x)$ is a linear combination of $\alpha^{(k)}(\sin x)$, $1 \leq k \leq n$ with coefficients being linear combination of sine and cosine functions with constant coefficients and since derivatives of α are bounded and sine and cosine are bounded functions we conclude that for each n there exists a constant M_n such that $|u^{(n)}(x)| < M_n$ for all x and $u^{(n)}(0) = 0$ for all n . Verifying other parts of conditions of the example 14 is easy. ■

Another evidence for strangeness of the series $\hat{f}(t)$ comes from the following argument. Suppose $f(t) = \sum_{m=-\infty}^{\infty} c_m e^{i\omega_m t}$. Then $f^{(n)}(t) = \sum (i\omega_m)^n c_m e^{i\omega_m t}$. Thus

$$\begin{aligned} \hat{f}(t) &= \sum_n \sum_m \frac{(-1)^n (i\omega_m)^n}{n!} c_m e^{i\omega_m t} t^n \\ &= \sum_m \sum_n \left(\frac{(-1)^n (i\omega_m)^n}{n!} t^n \right) c_m e^{i\omega_m t} \\ &= \sum_m e^{-i\omega_m t} c_m e^{i\omega_m t} \\ &= \sum_{m=-\infty}^{\infty} c_m \\ &= f(0). \end{aligned}$$

But the above argument is again analytically ill, since we have exchanged the order of two infinite sums which from the theorems of mathematical analysis we are not allowed in general to do so. In order to be able to do so there exists a general theorem on double series which states that if for a double infinite series $\sum_{m,n} a_{mn}$ for each n the series $\sum_m |a_{mn}|$ is convergent which we show its sum by b_n and the series $\sum_n b_n$ is also convergent then we are allowed to exchange the order of summation. That is we have $\sum_n \sum_m a_{mn} = \sum_m \sum_n a_{mn}$. Now let us check if this criterion can be applied to the double series whose entries are $a_{mn} := \frac{(-im)^n}{n!} c_m e^{imt} t^n$. For simplicity we have assumed that $\omega_m = 1$ for all m . We have $\sum_m |a_{mn}| = 2 \frac{t^n}{n!} \sum_{m=1}^{\infty} m^n |c_m|$. But in Fourier analysis it is well known that the series $f^{(n)}(t) = \sum (im)^n c_m e^{imt}$ is absolutely convergent. That is the series $\sum_{m=1}^{\infty} m^n |c_m|$ is convergent which we show its sum by α_n . Thus we have $\sum_m |a_{mn}| = 2 \frac{\alpha_n t^n}{n!}$. Thus we should verify the convergence of the series $\sum_{n=0}^{\infty} \frac{\alpha_n t^n}{n!}$. But this is a power series whose convergence radius is given by $R = \alpha^{-1}$ where $\alpha := \limsup \sqrt[n]{\frac{\alpha_n}{n!}} = \limsup \sqrt[n]{\frac{\alpha_n}{n!}}$. The point is that we are not sure if $R \neq 0$? See the following examples, [5].

Example 17 *The function*

$$f(t) := \sum_{m=0}^{\infty} \frac{1}{m!} e^{i2^m t} \quad (2.13)$$

is smooth nowhere analytic, in the sense that convergence radius of the Taylor's series of f at each point is zero and therefore $\hat{f}(t)$ diverges for all $t \neq 0$.

Proof Since for all n we have $\sum_{m=0}^N \left| \frac{(2^m i)^n}{m!} e^{i2^m t} \right| \leq \sum_{m=0}^N \frac{(2^m)^n}{m!}$ and since the later series converges to $i^n e^{2^n}$, we conclude that f and its derivatives $f^{(n)}$ are well defined [2]. Next we have $f^{(n)}(0) = \sum_{m=0}^{\infty} \frac{(2^m i)^n}{m!} = i^n e^{2^n}$. Thus $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n = \sum_{n=0}^{\infty} \frac{i^n e^{2^n}}{n!} t^n$. The convergence radius is obtained by the ratio test as follows. $\left| \frac{i^{n+1} e^{2^{n+1}}}{(n+1)!} \right| = \frac{e^{2^{n+1}}}{n+1} \rightarrow \infty$. Thus the radius of convergence of the Taylor series of f at $t = 0$ is zero. Now since f is periodic with period π we conclude that the radius of convergence of the Taylor series of f at $t = k\pi$ is zero for all $k \in \mathbb{Z}$.

Next for any integer N we set $g_N(t) = \sum_{m=0}^N \frac{1}{m!} e^{i2^m t}$ and $h_N(t) = \sum_{m=N+1}^{\infty} \frac{1}{m!} e^{i2^m t}$. We have $f = g_N + h_N$. Clearly g_N is everywhere analytic. h_N is periodic with period $\frac{\pi}{2^N}$. Thus by a similar argument as above we conclude that the radius of convergence of the Taylor series of h_N at $t = \frac{k\pi}{2^N}$ is zero for all $k \in \mathbb{Z}$. Thus the radius of convergence of the Taylor series of f at $t = \frac{k\pi}{2^N}$ is zero for all $k \in \mathbb{Z}$, too. Since the set of all numbers $\frac{k\pi}{2^N}, k \in \mathbb{Z}, N \in \mathbb{N}$ is dense in \mathbb{R} , we conclude that the radius of convergence of the Taylor series of f at any $t \in \mathbb{R}$ is zero. Thus by the theorem 12 part (i), $\hat{f}(t)$ diverges for all $t \neq 0$. ■

Example 18 *The function*

$$f(t) := \sum_{m=1}^{\infty} \frac{1}{m!} e^{i2^{-m} t} \quad (2.14)$$

is analytic at $t = 0$ whose convergence radius is infinity. Thus $\hat{f}(t)$ converges for all t to $f(0)$.

Proof The proof is similar to the previous example. ■

Open Questions 1. For the function

$$f(t) := e^{\frac{-1}{t^2}}, \quad t \neq 0, \quad f(0) = 0,$$

does $\hat{f}(t)$ converge?

2. Is there non-analytic functions f such that the series \hat{f} is point-wise or uniformly convergent and among such functions if there is any, is there any function such that the sum of the series is non-constant?

3. Verify the convergence of $\hat{f}(t)$ in the example 14 for $t \neq \frac{(2m+1)\ell}{2^n}$.

4. If we define a linear differential operator of infinite order

$$f \mapsto f(0) - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n f^{(n)}(t) \quad (2.15)$$

then in above we showed that analytic functions around origin are contained in the space of eigenfunctions of the zero eigenvalue of this operator. Now the question arises that: are there nonzero eigenvalues for this operator?

References

- [1] C.E. Burkhardt and J.J. Leventhal, *Foundations of quantum Physics*, Springer-Verlag 2008.
- [2] Tom M. Apostol *Mathematical Analysis*, Addison-Wesley, 1975
- [3] S. Kim and K. Kwon, *Smooth (C^∞) but nowhere Analytic Functions*, Monthly, **107** (2002) 264-266.
- [4] R.L. Bishop and R.J. Crittenden, *Geometry of Manifolds*, Academic Press (American Mathematical Society) 2001.
- [5] D. Rusin, <http://www.math.niu.edu/~rusin/known-math/99/nowhere-analy>